MADHAVA MATHEMATICS COMPETITION

(A Mathematics Competition for Undergraduate Students)

Organized by

Department of Mathematics, S. P. College, Pune

and

Homi Bhabha Centre for Science Education, T.I.F.R., Mumbai

Date : 4 / 1 / 2015 **Max. Marks : 100**

Time : 12.00 noon to 3.00 p.m.

N. B. : Part I carries 20 marks, Part II carries 30 marks and Part III carries 50 marks.

Part I

N.B. Each question in Part I carries 2 marks.

1. How many five digit positive integers that are divisible by 3 can be formed using the digits 0, 1, 2, 3, 4 and 5, without any of the digits getting repeated?
   A) 216 B) 96 C) 120 D) 625 .

2. If $I = \int_0^1 \frac{1}{1+x^8}\, dx$, then
   A) $I < \frac{1}{2}$ B) $I < \frac{\pi}{4}$ C) $I > \frac{\pi}{4}$ D) $I = \frac{\pi}{4}$.

3. Find a and b so that $y = ax + b$ is a tangent line to the curve $y = x^2 + 3x + 2$ at $x = 3$.
   A) $a = 9, b = -7$ B) $a = 3, b = -2$ C) $a = -9, b = 7$ D) $a = -3, b = 2$.

4. Suppose p is a prime number. The possible values of gcd of $p^3 + p^2 + p + 11$ and $p^2 + 1$ are
   A) 1,2,5 B) 2,5,10 C) 1,5,10 D) 1,2,10.

5. Consider all 2×2 matrices whose entries are distinct and belong to $\{1, 2, 3, 4\}$. The sum of determinants of all such matrices is
   A) 4! B) 0 C) negative D) odd.

6. Choose the correct alternative:

   A) The Taylor series of $\sin\frac{1}{x}$ about $x = \frac{2}{\pi}$ does not exist.

   B) The coefficient of $\left(x - \frac{2}{\pi}\right)^2$ in the Taylor series of $\sin\frac{1}{x}$ about $x = \frac{2}{\pi}$ is $\frac{-\pi^4}{32}$.

   C) The Taylor series of $\sin\frac{1}{x}$ about $x = \frac{2}{\pi}$ has negative powers of x.

   D) The coefficient of $\left(x - \frac{2}{\pi}\right)^2$ in the Taylor series of $\sin\frac{1}{x}$ about $x = \frac{2}{\pi}$ is 0.

7. Consider all right circular cylinders for which the sum of the height and circumference of the base is 30 cm. The radius of the one with maximum volume is
   A) 3 B) 10 C) $\frac{10}{\pi}$ D) $\frac{\pi}{10}$.

8. In how many ways can you express $2^3 3^5 5^7 7^{11}$ as a product of two numbers, ab, where $\gcd(a, b) = 1$ and $1 < a < b$?
   A) 5 B) 6 C) 7 D) 8.

9. The value of $\int_a^b \sin x\, dx$ is
   A) $(b - a)\sin c$ B) $(b - a)\cos c$ C) $\frac{\sin c}{b - a}$ D) $\frac{\cos c}{b - a}$

   for some real number c such that $a \leq c \leq b$.

10. Suppose a, b, c are three distinct integers from 2 to 10 (both inclusive). Exactly one of ab, bc and ca is odd and abc is a multiple of 4. The arithmetic mean of a and b is an integer and so is the arithmetic mean of a, b and c. How many such (unordered) triplets are possible?
    A) 4 B) 5 C) 6 D) 7.

Part II

N.B. Each question in Part II carries 6 marks.

1. Let $P(x) = \sum_{r=0}^{n} c_r x^r$ be a polynomial with real coefficients with $c_0 > 0$ and $\sum_{r=0}^{[n/2]} \frac{c_{2r}}{2r+1} < 0$. Prove that P has root in $(-1, 1)$.

2. If $|z_1| = |z_2| = |z_3| > 0$ and $z_1 + z_2 + z_3 = 0$, then show that the points representing the complex numbers z_1, z_2, z_3 form an equilateral triangle.

3. If $1, \alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are n^{th} roots of unity, prove that
$$\frac{1}{2-\alpha_1} + \frac{1}{2-\alpha_2} + \cdots + \frac{1}{2-\alpha_{n-1}} = \frac{(n-2)2^{n-1}+1}{2^n - 1}.$$

4. Let $f(x)$ be a monic polynomial of degree 4 such that $f(1) = 10, f(2) = 20, f(3) = 30$. Find $f(12) + f(-8)$.

5. Find all solutions (a, b, c, n) in positve integers for the equation $2^n = a! + b! + c!$.

Part III

1. Suppose the polynomials f and g have the same roots and $\{x \in \mathbb{C} : f(x) = 2015\} = \{x \in \mathbb{C} : g(x) = 2015\}$, then show that $f = g$. [13]

2. Give an example of a function which is continuous at exactly two points and differentiable at exactly one of them. Justify your answer. [13]

3. Let A be any $m \times n$ matrix whose entries are positive inegers. A step consists of transforming the matrix either by multiplying every entry of *a row* by 2 or subtracting 1 from every entry of *a column*. Can you transform A into the zero matrix in finitely many steps? Justify your answer. [12]

4. Let S be the set of positive integers that do not have zero in their decimal representation. Thus $S = \{1, 2, 3, \cdots, 9, 11, 12, \cdots, 19, 21, \cdots, 99, 111, \cdots\}$.
   Show that the series $\sum_{n \in S} \frac{1}{n}$ converges. [12]