MADHAVA MATHEMATICS COMPETITION

Date: January 5, 2014 Time: 12 noon- 3 p.m.

Part I

N.B. Each question in Part I carries 2 marks. [20]

1. If $x^3 - x + 1 = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3$, then (a_0, a_1, a_2, a_3) equals

   A. $(1, -1, 0, 1)$ **B.** $(7, 6, 10, 1)$ **C.** $(7, 11, 12, 6)$ **D.** $(7, 11, 6, 1)$

2. Suppose $f(x)$ and $g(x)$ are real-valued differentiable functions such that $f'(x) \geq g'(x)$ for all x in $[0, 1]$. Which of the following is necessarily true?

   A. $f(1) \geq g(1)$ **C.** $f(1) - g(1) \geq f(0) - g(0)$

   B. $f - g$ has no maximum on $[0, 1]$ **D.** $f + g$ is a non-decreasing function on $[0, 1]$

3. The equation $x^4 + x^2 - 1 = 0$ has

   A. two positive and two negative roots **C.** one positive, one negative and two non-real roots

   B. all positive roots **D.** no real root

4. Let n be a natural number. Let A and B be $n \times n$ matrices. If A is invertible, then which of the following is necessarily true?

   A. $rank(AB) < rank(B)$ **C.** $rank(AB) = rank(B)$

   B. $rank(AB) > rank(B)$ **D.** $rank(AB) < rank(A)$

5. Let X be a set and A, B, C be its subsets. Which of the following is necessarily true?

   A. $A - (A - B) = B$ **C.** $A - (B \cup C) = (A - B) \cup (A - C)$

   B. $A - (B \cap C) = (A - B) \cap (A - C)$ **D.** $B - (A - B) = B$

6. For a real number x we let $[x]$ denote the largest integer not exceeding x. For a natural number n, let $a_n = \frac{[n\sqrt{2}]}{n}$. The limit $\lim\limits_{n\to\infty} a_n$

   A. equals 0 **B.** equals $\left[\sqrt{2}\right]$ **C.** equals $\sqrt{2}$ **D.** does not exist

7. Let M be a two-digit natural number. Let N be the natural number whose digits are that of M but are in the reverse order. Which of the following CANNOT be the sum of M and N?

   A. 181 **B.** 165 **C.** 121 **D.** 154

8. The value of $\lim\limits_{x\to 1} \dfrac{\int_1^x e^{t^2} dt}{x^2 - 1}$ is

   A. 0 **B.** 1 **C.** e **D.** $e/2$

9. Let n be any positive integer and $1 \leq x_1 < x_2 < \cdots < x_{n+1} \leq 2n$, where each x_i is an integer. Which of the following must be true?

   (I) There is an i such that x_i is a square of an integer.

   (II) There is an i such that $x_{i+1} = x_i + 1$.

   (III) There is an i such that x_i is prime.

   A. I only **B.** II only **C.** I and II only **D.** II and III only

10. Two real numbers x and y are chosen uniformly at random from the interval $[0, 1]$. Find the probability that $2x > y$.

    A. 1/4 **B.** 1/2 **C.** 2/3 **D.** 3/4

Part II

N.B. Each question in Part II carries 6 marks. [30]

1. Let A be an 8×3 matrix in which every entry is either 1 or -1, and no two rows are identical. Find the rank of A.

2. Find all pairs (x, y) of integers such that $y^2 = x(x+1)(x+2)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is a decreasing function. If a, b, c are real numbers with $a < c < b$, prove that $(b-c)f(a) + (c-a)f(b) \leq (b-a)f(c)$.

4. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be a polynomial with integer coefficients such that a_0, a_3 and $f(1)$ are odd. Show that f has no rational root.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x-y) = f(x)f(y)$ and $f(x) \neq 0$ for all x. Find $f(3)$.

Part III

1. Prove that the equation $e^x - ln(x) - 2^{2014} = 0$ has exactly two positive real roots. [12]

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a non-constant function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that

   (a) $f(x) \neq 0$ for all $x \in \mathbb{R}$;

   (b) $f(x) > 0$ for all $x \in \mathbb{R}$;

   (c) if f is differentiable at 0, then f is differentiable on $\mathbb{R}$ and there exists some real number β such that $f(x) = \beta^x$ for all $x \in \mathbb{R}$. [12]

3. Let n be a natural number. Suppose $P_1, P_2, \cdots, P_n$ are points on a circle of radius 1. Prove that

$$\sum_{1 \leq i < j \leq n} d(P_i, P_j)^2 \leq n^2 \,,$$

   where for points X and Y in the plane, we denote by $d(X, Y)$ the distance between them. Prove that equality can hold for every natural number n. [13]

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that $f(0) = 0$. Suppose that $|f(z) - f(w)| = |z - w|$ for any $w \in \{0, 1, i\}$ and $z \in \mathbb{C}$. Prove that $f(z) = \alpha z$ or $f(z) = \alpha \overline{z}$ for some $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. [13]