Solutions to Madhava Mathematics Competition 2011
Part I
N.B. Each question in Part I carries 2 marks.

1. If $N = 1! + 2! + 3! + \cdots + 2011!$, then the digit in the units place of the number N is
(a) 1 (b) 3 (c) 0 (d) 9.
Answer : (b)

   Note that $5! \equiv 0 \pmod{10}$. Thus, $1! + 2! + 3! + 4! = 33 \equiv 3 \pmod{10}$.

2. The set of all points z in the complex plane satisfying $z^2 = |z|^2$ is a
(a) pair of points (b) circle (c) union of lines (d) line.
Answer : (d)

   $z = 0$ and if $z \neq 0$ then $zz = z\bar{z}$. Hence, $z = \bar{z}$. Hence, imaginary part of $z = 0$.

3. If the arithmetic mean of two numbers is 26 and their geometric mean is 10, then the equation with these two numbers as roots is
(a) $x^2 + 52x + 100 = 0$ (b) $x^2 - 52x - 100 = 0$
(c) $x^2 - 52x + 100 = 0$ (d) $x^2 + 52x - 10 = 0$.
Answer : (c)

   If the roots are α and β then $\alpha + \beta = 26$ and $\alpha\beta = 100$.

4. All points lying inside the triangle with vertices at the points $(1, 3)$, $(5, 0)$ and $(-1, 2)$ satisfy
(a) $3x + 2y \geq 0$ (b) $2x + y - 13 \geq 0$
(c) $2x - 3y - 12 \geq 0$ (d) $-2x + y \geq 0$.
Answer : (a)

   Substitute the coordinates of the points.

5. For $n \geq 3$, let A be an $n \times n$ matrix. If rank of A is $n - 2$, then rank of adjoint of A is
(a) $n - 2$ (b) 2 (c) 1 (d) 0.
Answer : (d)

   Rank of the matrix is $n-2$. Hence, every $(n-1) \times (n-1)$ minor equals 0. Hence, every entry of the adjoint of A is 0.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an odd and differentiable function. Then for every $x_0 \in \mathbb{R}$, $f'(-x_0)$ is equal to
(a) $f'(x_0)$ (b) $-f'(x_0)$ (c) 0 (d) None of these.
Answer : (a)

Use chain rule.

7. If $S = \{a, b, c\}$ and the relation R on the set S is given by $R = \{(a, b), (c, c)\}$, then R is
(a) reflexive and transitive (b) reflexive but not transitive
(c) not reflexive but transitive (d) neither reflexive nor transitive.
Answer : (c)

$(b, b) \notin R$ Hence, R is not reflexive. However, R is transitive.

8. Let $a_1 = 1,\ a_{n+1} = \left(\frac{1+n}{n}\right) a_n$ for $n \geq 1$. Then the sequence $\{a_n\}$ is
(a) divergent (b) decreasing (c) convergent (d) bounded.
Answer : (a)

Note that $a_n = n$ for every n. Hence, $< a_n >$ is an unbounded sequence. Hence, divergent.

9. The coefficient of x^{2n-2} in

$$f(x) = (x-1)(x+1)(x-2)(x+2)\cdots(x-n)(x+n)$$

is
(a) 0 (b) $\frac{-n(n+1)(2n+1)}{6}$ (c) $\frac{n(n+1)(2n+1)}{6}$ (d) $\frac{-n(n+1)}{2}$.
Answer : (b)

Note that $f(x) = (x^2-1)(x^2-2^2)\cdots(x^2-n^2)$. Hence the coefficient of x^{2n-2} is sum of squares of the numbers from 1 to n with negative sign.

10. The number of roots of $g(x) = 5x^4 - 4x + 1 = 0$ in $[0, 1]$ is
(a) 0 (b) 1 (c) 2 (d) 3.
Answer : (c)

$g(0) > 0$ and $g(1) > 0$ while $g(1/2) < 0$. Hence, $g(x) = 0$ has at least two roots. Note that $g'(x) < 0$ if $x^3 < 1/5$ and $g'(x) > 0$ if $x^3 > 1/5$. Hence, the function is decreasing in $(-\infty, \sqrt[3]{1/5})$ and increasing on $(\sqrt[3]{1/5}, \infty)$. At $\sqrt[3]{1/5}$ the function has absolute minimum.

Part II

N.B. Each question in Part II carries 5 marks.

1. If $n \geq 3$ is an integer and k is a real number, prove that n is equal to the sum of n^{th} powers of the roots of the equation $x^n - kx - 1 = 0$.
   Solution :
   Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the equation $x^n - kx - 1 = 0$. Therefore $\alpha_i^n = k\alpha_i + 1,\ 1 \leq i \leq n.$ [1 mark]

   Therefore $\sum_{i=1}^{n} \alpha_i^n = k \sum_{i=1}^{n} \alpha_i + n,\ 1 \leq i \leq n.$ [2 marks]

   But as $n \geq 3$, $\sum_{i=1}^{n} \alpha_i = 0$, since the coefficient of x^{n-1} is zero. Thus $\sum_{i=1}^{n} \alpha_i^n = n.$ [2 marks]

2. Find all positive integers n such that $(n2^n - 1)$ is divisible by 3.
   Solution : Note that $2^2 \equiv 1 \pmod 3$. Hence, $2^{2k} \equiv 1 \pmod 3$. Thus if n is even then $2^n \equiv 1 \pmod 3$ and if n is odd then $2^n \equiv 2 \pmod 3$. Hence, if n is even then $n2^n - 1 \equiv (n-1) \pmod 3$ and if n is odd then $n2^n - 1 \equiv 2n - 1 \pmod 3$. $3|(n2^n - 1)$ Hence $(n-1) \equiv 0 \pmod 3$ if n is even and $3|(-n-1)$ if n is odd. Hence, $n = 6k + 4$ [2 marks]
   or $n = 6k + 5$. [2 marks]

   Further, if $n = 6k + 4$ or $n = 6k + 5$ then $n|n2^n - 1$. [1 mark]

3. Start with the set $S = \{3, 4, 12\}$. At any stage you may perform the following operation: Choose any two elements $a, b \in S$ and replace them by $\left(\frac{3a - 4b}{5}\right)$ and $\left(\frac{4a + 3b}{5}\right)$. Is it possible to transform the set S into the set $\{4, 6, 12\}$ by performing the above operation a finite number of times?
   Solution :
   When we replace a and b by $a_1 = \left(\frac{3a - 4b}{5}\right)$ and $b_1 = \left(\frac{4a + 3b}{5}\right)$. The set $\{a, b, c\}$ changes to $\{a_1 = \left(\frac{3a - 4b}{5}\right), b_1 = \left(\frac{4a + 3b}{5}\right), c_1 = c\}$. The sum of squares of the elements of this set is

$\left(\frac{3a-4b}{5}\right)^2 + \left(\frac{4a+3b}{5}\right)^2 + c^2 = a^2 + b^2 + c^2$. Thus the new set $\{a_1, b_1, c_1\}$ satisfies the condition $a^2 + b^2 + c^2 = a_1^2 + b_1^2 + c_1^2$. Now the set $\{3, 4, 12\}$ has sum of squares equal to 169, where as the new set $\{4, 6, 12\}$ has sum of squares equal to 196. The two sums are different.

Hence it is not possible to transform the set $\{3, 4, 5\}$ to $\{4, 6, 12\}$. [5 marks]

Note: If the answer is no by trial and error then give 1 mark.

4. Let $a < b$. Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Let α be a real number. If $f(a) = f(b) = 0$, show that there exists $x_0 \in (a, b)$ such that $\alpha f(x_0) + f'(x_0) = 0$.
   Solution :
   Define $g(x) = e^{\alpha x} f(x)$. Then $g'(x) = e^{\alpha x}[\alpha f(x) + f'(x)]$. [2 marks]

   As $f(a) = f(b) = 0$, we have $g(a) = g(b) = 0$. By Rolle's theorem, there exists $x_0 \in (a, b)$ such that $g'(x_0) = 0$. This implies $\alpha f(x_0) + f'(x_0) = 0$. [3 marks]

Part III

N.B. Each question in Part III carries 12 marks.

1. Let M_n be the $n \times n$ matrix with all 1's along the main diagonal, directly above the main diagonal and directly below the main diagonal and 0's everywhere else. For example,

$$M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \; M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \quad \text{Let } d_n = \det M_n.$$

   (a) Find d_1, d_2, d_3, d_4. [If all are done 2 marks]
   (b) Find a formula expressing d_n in terms of d_{n-1} and d_{n-2}, for all $n \geq 3$. [3 for expressing it and 3 for the proof.]
   (c) Find d_{100}. [4 marks]
   Solution :

   - Note, $M_1 = 1$, so $\det(M_1) = d_1 = 1$.
   - $M_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, so clearly $\det(M_2) = d_2 = 0$.

- Next, $M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, so clearly $\det(M_3) = 1(\det(M_2)) - 1(1) = d_3 = -1$.
- Note $M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $\det(M_4) = 1(\det(M_3)) - 1(\det(M_2)) = d_4 = -1$.

  (Some students will realize induction here and will straightaway go to general formula).
- Let $M_n = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$.

  Then, we claim: $\det(M_n) = 1(\det(M_{n-1})) - 1(\det(M_{n-2}))$ i.e.,

$$\det(M_n) = \det(M_{n-1}) - \det(M_{n-2}).$$

- The proof follows from the row-expansion formula for the determinant.

  Expanding along the first row, in M_n, we get:

$$\det(M_n) = 1(\det(M_{n-1})) - 1(\det K),$$

  where K is the following $(n-1) \times (n-1)$ matrix:

$$K = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & \cdots & 1 & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

  Again expanding along the first row, note that

$$(\det K) = 1(\det(M_{n-2})) - 1(\det K'),$$

where $K' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

Clearly, $\det K' = 0$, as all the entries in one column of K' are 0. This proves the claim.

- Now, it is easy to find that:

$$d_5 = d_4 - d_3 = (-1) - (-1) = 0,$$
$$d_6 = d_5 - d_4 = 0 - (-1) = 1,$$
$$d_7 = d_6 - d_5 = 1 - 0 = 1,$$
$$d_8 = d_7 - d_6 = 1 - 1 = 0.$$

- In fact, finding a few more terms makes the pattern obvious by looking at the following table:
  $d_1 = 1 = d_7$
  $d_2 = 0 = d_8$
  $d_3 = -1 = d_9$
  $d_4 = -1 = d_{10}$
  $d_5 = 0 = d_{11}$
  $d_6 = 1 = d_{12}$

- Thus, we can calculate d_n for any n by the following formula:

$$d_n = 1, \text{if } n \equiv 0, 1 \pmod 6,$$
$$d_n = 0, \text{if } n \equiv 2, 5 \pmod 6,$$
$$d_n = -1, \text{if } n \equiv 3, 4 \pmod 6.$$

2. Let $p(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - 4x^{2n-3} + \cdots - 2nx + (2n+1)$.
   Show that the polynomial $p(x)$ has no real root.
   Solution :
   If $x \leq 0$ then $p(x) > 0$. [2 marks]
   Let $x > 0$.
   $p(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - 4x^{2n-3} + \cdots - 2nx + (2n+1)$.
   $xp(x) = x^{2n+1} - 2x^{2n} + 3x^{2n-1} - 4x^{2n-2} + \cdots - 2nx^2 + (2n+1)x$.

$xp(x) + p(x) = x^{2n+1} - x^{2n} + x^{2n-1} - x^{2n-2} + \cdots + x + (2n+1)$.
$(1+x)p(x) = x\left(\dfrac{1+x^{2n+1}}{1+x}\right) + (2n+1)$.
$\Rightarrow p(x) > 0$ for $x > 0$. [10 marks]

Note: If done for an interval then maximum 2 marks.

3. Let $f(x) = x^{10} + a_1x^9 + a_2x^8 + \cdots + a_{10}$ where a_i's are integers.

   If all the roots of $f(x)$ are from the set $\{1, 2, 3\}$, determine the number of such polynomials. Further, if $g(x)$ is the sum of all such polynomials $f(x)$, then show that the constant term of $g(x)$ is $\frac{1}{2}(3^{12}+1) - 2^{12}$.

   Solution :
   Note that $f(x) = (x-1)^a(x-2)^b(x-3)^c$, where $a + b + c = 10$ and $a, b, c \geq 0$ are integers. [2 marks]

   If $c = 0$ then there are 11 solutions. If $c = 1$ then there are 10 solutions. If $c = 2$ then there are 9 solutions. If $c = 3$ then there are 8 solutions and so on. Thus for every c there are $11 - c$ solutions. Hence, the total number of solutions is $1 + 2 + \cdots + 11 = 66$. Hence, there are 66 such polynomials. [4 marks]

   The constant term of $g(x)$ i.e a_{10} equals

$$
\begin{aligned}
& \sum_{a+b+c=10} 1^a 2^b 3^c \qquad \text{[2 marks]} \\
= \;& 3^{10} + 3^9(2+1) + 3^8(2^2 + 2(1) + 1^2) + \cdots + 3^0(2^{10} + 2^9 + \cdots + 1) \\
= \;& 3^{10}(2-1) + 3^9(2^2-1)) + 3^8(2^3-1) + \cdots + 3^0(2^{11}-1) \\
= \;& 3^{11}2(1 + \frac{2}{3} + \cdots + \left(\frac{2}{3}\right)^{10}) - \frac{1}{2}(3^{11}-1) \\
= \;& 3^{11}2(1 - (\frac{2}{3})^{11}) - \frac{1}{2}(3^{11}-1) \\
= \;& 2(3^{11} - 2^{11}) - \frac{1}{2}(3^{11}-1) \qquad \text{[4 marks]}
\end{aligned}
$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f(x+h) - f(x) = hf'(x + \frac{1}{2}h),$$

for all real x and h. Prove that f is a polynomial of degree at most 2.
Solution :
From the given condition, we have $f(x+h)-f(x-h) = 2hf'(x),\ \forall x, h$. Therefore putting $x = 0$, we get $f(h) - f(h) = 2hf'(0),\ \forall h$. Differentiating with respect to h, $f'(h) + f'(-h) = 2f'(0),\ \forall h$.
Now define $g(x) = f'(x) - f'(0)$. Then $g(0) = 0$ and $g(-x) = -g(x)$.
Again $f'(a + h) + f'(a - h) = 2f'(a)$. Putting $a + h = x, a - h = y$ in above expression, we get $f'(x) + f'(y) = 2f'(\frac{x+y}{2})$ (*)
and putting $h = a$, we get $f'(2a) + f'(0) = 2f'(a)$.
Therefore $f'(a) + f'(0) = 2f'(\frac{a}{2})$. (**)
From (*) $f'(x) + f'(y) = 2f'(\frac{x+y}{2})$
From (**) $f'(x) + f'(y) = f'(x + y) + f'(0)$.
$g(x+y) = f'(x+y) - f'(0) = f'(x) + f'(y) - 2f'(0) = [f'(x) - f'(0)] + [f'(y) - f'(0)] = g(x) + g(y)$.
Therefore $g(kx) = kg(x)$, $g(\frac{x}{n}) = \frac{1}{n}g(x)$, $g(\frac{m}{n}x) = \frac{m}{n}g(x)$.
Now g is continuous, $g(\alpha x) = \alpha g(x),\ \forall \alpha \in \mathbb{R}$. Therefore g is linear. Therefore $f'(x) - f'(0) = ax$. Therefore $f'(x) = f'(0) + ax$. Therefore $f(x) = \frac{a}{2}x^2 + f'(0)x + c$.

defining the function g is crucial hence 8 marks for that and then 4 marks for showing that it is linear.

5. (a) Let $n = 9$. Express n as a sum of positive integers such that their product is maximum. Find the value of the maximum product.
(b) Repeat part (a) for $n = 10$ and $n = 11$.
(c) Given a positive integer $n \geq 6$, express n as a sum of positive integers such that their product is maximum. Find the value of the maximum product.
Solution :
(a) 9=3+3+3.
(b) 10=2+2+3+3, 11=2+3+3+3.

Note that expressing 9, 10 and 11 carries 1 mark each.

(c) Let $n \geq 5$. Note that $n = (n - 3) + 3$ and $n < 3(n - 3)$. [6 marks]

Hence, we write n as sum of 3's till we get a number < 5. If the resulting number is 4, then we express it as $2 + 2$. If it is 2 or 3 then keep it as it is. [3 marks]