Madhava Mathematics Competition January 6, 2013

Solutions and scheme of marking

Part I

N.B. Each question in Part I carries 2 marks.

1. If $p(x)$ is a non-constant polynomial, then $\lim\limits_{k\to\infty} \dfrac{p(k+1)}{p(k)}$ is equal to

   (a) 1 (b) 0 (c) -1 (d) the leading coefficient of $p(x)$.

   Solution: (a)

   It is sufficient to find the limit in case of $p(x) = ax + b$.

   $\lim\limits_{k\to\infty} \dfrac{p(k+1)}{p(k)} = 1.$

2. The number of continuous functions f from $[-1, 1]$ to $\mathbb{R}$ satisfying $(f(x))^2 = x^2$ for all $x \in [-1, 1]$ is

   (a) 2 (b) 3 (c) 4 (d) infinite.

   Solution: (c)

   The expression $(f(x))^2 = x^2$ implies $f(x) = \pm x$ or $f(x) = \pm|x|$.

3. Let $q \in \mathbb{N}$. The number of elements in set $\{(\cos\frac{\pi}{q} + i \sin\frac{\pi}{q})^n \mid n \in \mathbb{N}\}$ is

   (a) 1 (b) q (c) infinite (d) $2q$.

   Solution: (d)

   The number of elements in the given set is equal to the number of q^{th} roots of $\{\cos n\pi + i \sin n\pi \mid n \in \mathbb{N}\} = \{\pm 1\}$ which are $2q$ in number since the q^{th} roots of 1 are distinct from the q^{th} roots of -1.

4. If $f(x) = |x|^{\frac{3}{2}}$, $\forall x \in \mathbb{R}$, then at $x = 0$,

   (a) f is not continuous (b) f is continuous but not differentiable (c) f is differentiable but f' is not continuous (d) f is differentiable and f' is continuous.

   Solution: (d)

   $$f'(x) = \frac{3}{2}\sqrt{x}, \text{ if x} > 0$$
   $$= -\frac{3}{2}\sqrt{|x|}, \text{ if x} < 0$$
   $$f'(0) = \lim_{h\to 0} \frac{|h|^{\frac{3}{2}}}{h} = 0.$$

   Therefore f' exists and is continuous at 0.

5. If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are roots of the equation $x^4 + x^3 + 1 = 0$, then the value of $(1 - 2\alpha_1)(1 - 2\alpha_2)(1 - 2\alpha_3)(1 - 2\alpha_4)$ is equal to

   (a) 19 (b) 16 (c) 15 (d) 20.

   Solution: (a)

   $(1 - 2\alpha_1)(1 - 2\alpha_2)(1 - 2\alpha_3)(1 - 2\alpha_4) = 1 - 2\sum \alpha_i + 4\sum \alpha_i\alpha_j - 8\sum \alpha_i\alpha_j\alpha_k + 16\alpha_1\alpha_2\alpha_3\alpha_4 = 1 - 2(-1) + 4(0) - 8(0) + 16(1) = 19.$

6. If A and B are 3×3 real matrices with rank $(AB) = 1$, then rank (BA) cannot be

   (a) 0 (b) 1 (c) 2 (d) 3.

   Solution: (d)

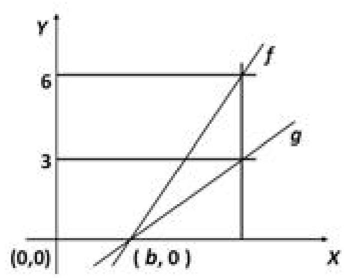
   If the rank of BA is 3, then the determinant of BA is nonzero. Then the determinant of AB is nonzero and hence the matrix AB is also invertible, which is not possible.

7. The number of common solutions of $x^{36} - 1 = 0$ and $x^{24} - 1 = 0$ in the set of complex numbers is
(a) 1 (b) 2 (c) 6 (d) 12.
Solution: (d)
The number of common solutions of $x^{36} - 1 = 0$ and $x^{24} - 1 = 0$ in the set of complex numbers is the $\gcd(36, 24) = 12$.

8. If f is a one to one function from $[0, 1]$ to $[0, 1]$, then
(a) f must be onto (b) f cannot be onto (c) $f([0, 1])$ must contain a rational number (d) $f([0, 1])$ must contain an irrational number.
Solution: (d)
Because f is one to one, $f([0, 1])$ and $[0, 1]$ are equivalent sets. Also $f([0, 1]) \subseteq [0, 1]$. Now $[0, 1]$ is uncountable and the set of rational numbers is countable. Hence $f([0, 1])$ must contain an irrational number.

9. There are 18 ways in which n identical balls can be grouped such that each group contains equal number of balls. Then the minimum value of n is
(a) 120 (b) 180 (c) 160 (d) 90.
Solution: (b)
The total number of required ways= the total number of factors of n.
$180 = 2^2 \times 3^2 \times 5$. Therefore the total number of factors of 180 is $3 \times 3 \times 2 = 18$.

10. Suppose f and g are two linear functions as shown in the figure. Then $\lim_{x \to b} \frac{f(x)}{g(x)}$
(a) is 2 (b) does not exist (c) is 3 (d) is $\frac{1}{2}$.



Solution: (a)
By L$'$ Hospital Rule,

$$\lim_{x \to b} \frac{f(x)}{g(x)} = \lim_{x \to b} \frac{f'(x)}{g'(x)} = \frac{6}{3} = 2.$$

Part II

N.B. Each question in Part II carries 5 marks. Attempt any FOUR:

(a) Let $A = (a_{ij})$ be $n \times n$ matrix, where $a_{ij} = \max\{i, j\}$. Find the determinant of A.
Solution: Let $a_{ij} = \max\{i, j\}$.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 2 & 3 & 4 & \dots & n \\ 3 & 3 & 3 & 4 & \dots & n \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ n & n & n & n & \dots & n \end{pmatrix}$$

Performing the operations $R_n - R_{n-1},\ R_{n-1} - R_{n-2}, \cdots, R_2 - R_1$ on A we get

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 \end{pmatrix}. \qquad [1]$$

Hence the determinant of A is $(-1)^{n+1}n$.

(b) Assume that f is a continuous function from $[0,2]$ to $\mathbb{R}$ and $f(0) = f(2)$. Prove that there exist x_1 and x_2 in $[0,2]$ such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$.

Solution: Define $g(x) = f(x+1) - f(x)$ for $x \in [0,1]$.

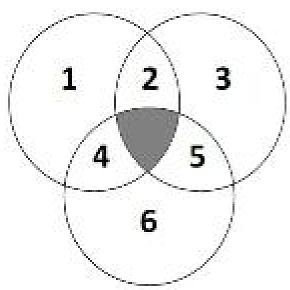
Note that $g(1) = f(2) - f(1)$ and $g(0) = f(1) - f(0) = -g(1)$.

Hence by Intermediate Value Property, there exists $x_0 \in [0,1]$ such that $g(x_0) = 0$.

Therefore $f(x_0 + 1) = f(x_0)$.

(c) Let $X = \{1, 2, 3, \cdots, 10\}$. Determine the number of ways of expressing X as $X = A_1 \cup A_2 \cup A_3$, where $A_1, A_2, A_3 \subseteq X$ and $A_1 \cap A_2 \cap A_3 = \phi$.

Solution: Each number from the set X has 6 choices as shown in the figure.



Therefore the total number of required ways is 6^{10}.

(d) For non-negative real numbers $a_1, a_2, \cdots, a_n$, show that $\frac{1}{n}\sum_{k=1}^{n} a_k e^{-a_k} \leq \frac{1}{e}$.

Solution: Let $f(x) = xe^{-x}$ for $x \geq 0$.

Then $f'(x) = e^{-x} - xe^{-x} = 0$ implies $e^{-x}(1-x) = 0$. Therefore $x = 1$ is the only critical point. Note that $f''(1) = -\frac{1}{e} < 0$.

Hence $f(1) = \frac{1}{e}$ is the maximum value of f.

Therefore $\frac{1}{n}\sum_{k=1}^{n} a_k e^{-a_k} \leq \frac{1}{n}\ n\ \frac{1}{e} = \frac{1}{e}$.

(e) Let A be a non-zero $1 \times n$ real matrix. Then show that the rank of A^tA is 1.

[Note that A^t denotes the transpose of the matrix A.]

Solution: Let $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$. Then

$$A^tA = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} = \begin{pmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2^2 & \cdots & a_2a_n \\ \vdots & \ddots & \ddots & \vdots \\ a_na_1 & a_na_2 & \cdots & a_n^2 \end{pmatrix}.$$

Without loss of generality we can assume that $a_1 \neq 0$. Performing row operations $R_2 \to R_2 - \frac{a_2}{a_1}R_1, R_3 \to R_3 - \frac{a_3}{a_1}R_1, \cdots, R_n \to R_n - \frac{a_n}{a_1}R_1$, we obtain

$$A^tA = \begin{pmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence the rank of A^tA is 1.

Part III

N.B. Each question in Part III carries 12 marks. Attempt any FIVE:

(a) Let $f(x) = \dfrac{(x-a)(x-b)}{(x-c)}, \quad x \neq c.$

Find the range of f in each of the following cases:

i] $a < c < b$ ii] $c < a < b$.

Solution: Let $f(x) = \dfrac{(x-a)(x-b)}{(x-c)} = y$.

Therefore $(x-a)(x-b) = y(x-c)$.

Therefore $x^2 - (a+b+y)x + (ab+yc) = 0$.

This quadratic equation has solution $x = \dfrac{(a+b+y) \pm \sqrt{(a+b+y)^2 - 4(ab+yc)}}{2}$.

This x is real if and only if the term in the square root is non-negative.

Therefore we need $(a+b+y)^2 - 4(ab+yc) \geq 0$.

Therefore $a^2 + b^2 + y^2 + 2ab + 2by + 2ay - 4ab - 4yc \geq 0$.

Therefore $y^2 + 2(a+b-2c)y + (a+b-2c)^2 + (a-b)^2 - (a+b-2c)^2 \geq 0$.

Therefore $[y + (a+b-2c)]^2 \geq (a+b-2c)^2 - (a-b)^2$.

Simplifying this we get, $[y + (a+b-2c)]^2 \geq 4(c-b)(c-a)$.

i] If $a < c < b$, then the product $(c-b)(c-a)$ is negative and the above inequality is true for all real values of y. Hence the range of f is $\mathbb{R}$.

ii] If $c < a < b$, then the product $(c-b)(c-a)$ is positive.

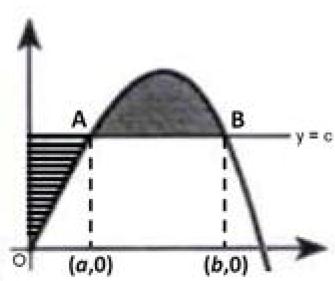
We have $y+(a+b-2c) \geq 2\sqrt{(c-b)(c-a)}$ or $y+(a+b-2c) \leq -2\sqrt{(c-b)(c-a)}$.

Therefore $y \geq (2c-a-b)+2\sqrt{(c-b)(c-a)}$ or $y \leq (2c-a-b)-2\sqrt{(c-b)(c-a)}$.

Hence the range of f is

$(-\infty, (2c-a-b) - 2\sqrt{(c-b)(c-a)}] \cup [(2c-a-b) + 2\sqrt{(c-b)(c-a)}, \infty)$

(b) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.



Solution: Suppose $A(a, c)$ and $B(b, c)$ are the points on the curve as shown in the figure. Therefore they satisfy $c = 2a - 3a^3 = 2b - 3b^3$.

The areas of the two shaded regions are equal if

$$ca - \int_0^a (2x - 3x^3)\, dx = \int_a^b (2x - 3x^3)\, dx - c(b-a).$$

Therefore $ca - [x^2 - 3\frac{x^4}{4}]_0^a = [x^2 - 3\frac{x^4}{4}]_a^b - cb + ca$

Therefore $ca - a^2 + 3\frac{a^4}{4} = b^2 - 3\frac{b^4}{4} - a^2 + 3\frac{a^4}{4} - cb + ca$.

Therefore $b^2 - 3\frac{b^4}{4} - bc = 0$.

Substituting the value of c we get, $b^2 - 3\frac{b^4}{4} - b(2b - 3b^3) = 0$.

Therefore $b^2 - 3\frac{b^4}{4} - 2b^2 + 3b^4 = 0$.

Therefore $\frac{9}{4}b^4 = b^2$.

Hence cancelling b^2 from both sides we get, $b^2 = \frac{4}{9}$.

Therefore $b = \frac{2}{3}$.

Substituting the value of b in the expression $c = 2b - 3b^3$ we get, $c = \frac{4}{9}$.

(c) Suppose $A_1, A_2, \cdots, A_n$ are vertices of a regular n-gon inscribed in a unit circle and P is any point on the unit circle. Prove that $\sum_{i=1}^{n} l(PA_i)^2$ is constant, where $l(PA_i)$ denotes the distance between P and A_i. [Hint: Use complex numbers.]

Solution: Here $n \geq 3$. Let $z_1, z_2, \cdots, z_n$ be the roots of unity.

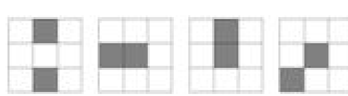
Therefore $\sum_{k=1}^{n} z_k = 0$.

$$\sum_{i=1}^{n} l(PA_i)^2 = \sum_{k=1}^{n} |z - z_k|^2 = \sum_{k=1}^{n} (z - z_k)\overline{(z - z_k)} = \sum_{k=1}^{n} (z - z_k)(\overline{z} - \overline{z_k}) =$$
$$\sum_{k=1}^{n} |z|^2 - \sum_{k=1}^{n} z\overline{z_k} - \sum_{k=1}^{n} z_k\overline{z} + \sum_{k=1}^{n} |z_k|^2 = n + n = 2n.$$

(d) A unit square of a chess board of size $n \times n$ gets infected if at least two of its neighbours are infected. Find the maximum number of infected unit squares if initially [i] 2 unit squares are infected, [ii] 3 unit squares are infected. Find the minimum number of unit squares that should be infected initially so that the whole chess board gets infected.

(Two unit squares are called neighbours if they share a common edge.)

Solution: If initially 2 unit squares are infected, then the maximum number of



infected unit squares is 4.

If initially 3 unit squares are infected, then the maximum number of infected unit squares is 9.

Observe that n diagonal infected squares can infect the whole chess board. Hence n are enough to infect the entire board.

To show that n are needed we observe that total perimeter of the infected region can not increase as the virus spreads over the board. When a new square is infected at least two of its boundary edges are absorbed into the infected region and at the most two boundary edges are added to it. Therefore if there are less than n squares that are initially infected the total perimeter of the infected region will be less than $4n$ initially and it will remain less than $4n$ as the virus spreads. Hence the entire board with the perimeter $4n$ will never be entirely infected.

(e) Let a, b, c be integers. Let $x = \frac{p}{q}, y = \frac{r}{s}$ be rational numbers satisfying $y^2 = x^3 + ax^2 + bx + c$. Show that there exists an integer t such that $q = t^2, s = t^3$.

Solution: Let $x = \frac{p}{q}, y = \frac{r}{s}$ be the rational solutions of $y^2 = x^3 + ax^2 + bx + c$ where $(p, q) = 1 = (r, s), q > 0, s > 0$.

Substituting the values of x and y, we get

$$\frac{r^2}{s^2} = \frac{p^3}{q^3} + \frac{ap^2}{q^2} + \frac{bp}{q} + c$$

This implies $r^2q^3 = p^3s^2 + ap^2qs^2 + bpq^2s^2 + cq^3s^2$ (*)

This implies $q|p^3s^2$. Therefore $q|s^2$. (1)

So, $q^2|ap^2qs^2$ implies $q^2|p^3s^2$ by (*)

Hence $q^2|s^2$. Therefore $q|s$. (2)

So, $q^3|ap^2qs^2$ and $q^3|bpq^2s^2$.

By (*) $q^3|p^3s^2$. Therefore $q^3|s^2$. (3)

Also (*) implies $s^2|r^2q^3$. Therefore $s^2|q^3$. (4)

By (3) and (4), $s^2 = q^3$.

But by (2), $q|s$. Therefore $s = qt$ for some $t \in \mathbb{N}$.

So, $s^2 = q^3$ implies $q^2t^2 = q^3$. Therefore $q = t^2$ and $s = qt = t^3$.

(f) Show that the polynomial $p_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ has no real root if n is even and exactly one real root if n is odd.

Solution: Observe that $p_n'(x) = p_{n-1}(x)$.

Note that $p_1(x) = 1 + x$ has exactly one real root $x = -1$.

Consider $p_2(x) = 1+x+\frac{x^2}{2!}$. The quadratic equation $p_2(x) = 0$ i.e. $x^2+2x+2 = 0$ has no real root.

Let $p_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. This is polynomial of odd degree. Therefore it has at least one real root. If $p_3(x)$ has two real roots a and b, then between a and b, there is a root of $p_3'(x) = p_2(x)$. But $p_2(x)$ has no real root. Hence $p_3(x)$ has exactly one real root.

Let $p_3(a) = 0$. Note that a is a negative real number. Observe that a is a critical point of $p_4(x)$. Now $p_4''(x) = p_3'(x) = p_2(x)$. Then $p_2(a) = p_3(a) - \frac{a^3}{3!} = -\frac{a^3}{3!} > 0$. Therefore $p_4''(a) > 0$. Hence $p_4(x)$ has minimum value at a.

Therefore $p_4(a) \leq p_4(x)$ for all x.

But $p_4(a) = p_3(a) + \frac{a^4}{4!} > 0$. Therefore $p_4(x) > 0$ for all x. Hence $p_4(x)$ has no real root. Repeating the similar argument we can prove the result.